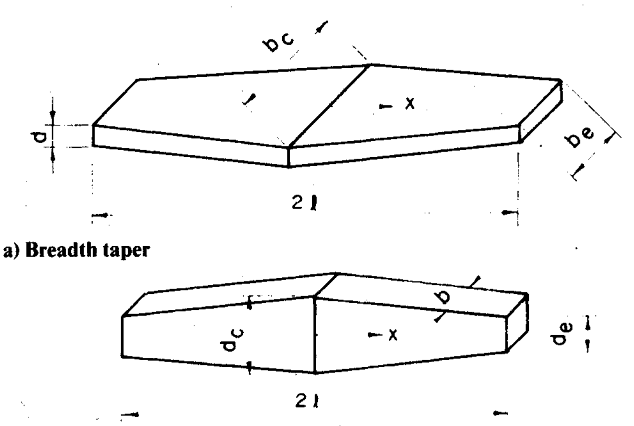

a) Breadth taper

b) Depth taper

Fig. 1 Geometry of tapered columns.

Trigonometric: $\lambda_L = \pi^2/4 = 2.4674$

$$\gamma = 1 + {}^1\!/\!_4\,(b/r_c)^2 \qquad (16)$$

Algebraic: $\lambda_L = 42/17 = 2.4706$

$$\gamma = 1 + 2.2600(b/r_c)^2 \qquad (17)$$

where r_c is the radius of gyration at the center of the beam. Note that λ_L is defined with respect to ℓ, the half length of the beam.

Numerical Results

The results for the uniform beam given in Eqs. (16) and (17) can be seen to be in good agreement when trigonometric and algebraic functions are used for the displacements. These results are also in good agreement with the finite element results presented earlier.[4]

Table 2 presents λ_L and γ values for both breadth- and depth-tapered columns for three values of β (0.0, 0.2, and 0.4). Again, the results from trigonometric and polynomial distributions are in close agreement. Further, it can be seen that for breadth-tapered columns, the linear buckling load decreases with increasing taper. The effect of nonlinearity (from γ values) can be seen to be relatively greater in the case of depth-tapered columns and it becomes pronounced when large depth tapers are considered. Also, the effect of nonlinearity decreases with increasing breadth taper, whereas the effect increases with increasing depth taper.

Conclusions

Rayleigh-Ritz solutions are presented for the linear thermal buckling load and thermal load ratios in the postbuckling range for simply supported tapered columns under thermal loading. The effect of nonlinearity on the thermal load ratios is found to be of the opposite tendency with increasing taper values in the breadth- and depth-tapered cases.

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Thermal Constriction Resistance with Arbitrary Heating in a Convectively Cooled Plate

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Nomenclature

a = contact length
a_q = thermal diffusivity
b = length of the plate
Bi = Biot number (hb/k)
c = thickness of the plate
e = eccentricity
Fo = Fourier number $(a_q t/b^2)$
h = heat-transfer coefficient
k = coefficient of thermal conductivity
L = characteristic length
P = parameter defined by Eq. (18)
q = heat flux
q^* = dimensionless heat flux
q_0 = constant value of dimensionless heat flux
R = thermal constriction resistance defined by Eq. (6)
R_c^* = dimensionless thermal constriction resistance
T = temperature distribution
T_f = ambient temperature
x,y = Cartesian coordinate system
α = aspect ratio (c/b)
ϵ = parameter defined by Eq. (21)
θ = dimensionless temperature $(T-T_b)/T_b$
θ_0 = initial value of dimensionless temperature
μ = characteristic root defined by Eq. (12)
ω = parameter defined by Eq. (21)

I. Introduction

TEMPERATURE control is one of the most important design problems in achieving high reliability in a launch vehicle or spacecraft. It requires a precise heat-transfer calculation and the designer is often called upon to predict the total thermal resistance present in the available heat path from the source to the sink. The thermal resistance developed through a constriction in the heat flow is usually high when compared to the resistance away from the constriction. The accurate prediction of thermal resistance requires a detailed transient analysis of a two-dimensional constriction. However, no such analysis has been reported in the literature. In contrast, steady-state analysis has been attempted by several authors. Veziroglu and Chandra[1] have provided a theoretical and experimental study of the subject for both symmetrical and eccentric constrictions. Schneider, Yovanovich, and Cane[2] have recently provided a steady-state analysis for a planar constriction with an arbitrary heat flux boundary condition. Mehta and Bose[3] have obtained the expression for the thermal constriction resistance of a large circular plate with the heat flux provided on a disk area.

The present work develops a theory for predicting the thermal constriction resistance of an eccentric contact area subjected to an arbitrary heat flux. A detailed transient

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analysis of a two-dimensional constriction is considered, and the analytical expressions for thermal constriction resistance are obtained for exponential and periodic types of heat fluxes.

II. Mathematical Formulation of the Thermal Problem

Consider the problem of transient heat transfer from an arbitrary contact area situated in the XOY plane as shown in Fig. 1a. The face parallel to the one that is subjected to the arbitrary heat flux boundary condition loses heat by Newtonian law. The other regions are perfectly insulated. The temperature field within this region must satisfy the energy equation, which is described for the two-dimension Cartesian geometry in dimensionless form as

$$\frac{\partial \theta}{\partial Fo} = \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} \tag{1}$$

The boundary conditions are prescribed as

$$\left.\frac{\partial \theta}{\partial x}\right|_{x=0,1} = 0 \tag{2}$$

and

$$\left[\frac{\partial \theta}{\partial y} + B_i \theta\right]_{y=\alpha} = 0 \tag{3}$$

and

$$\left.\frac{\partial \theta}{\partial y}\right|_{y=0} = -q^* U[x-e-½a(x)] + q^* U[x-e+½a(x)] \tag{4}$$

for eccentric contact, $U(x)$ is a unit step function with the properties, $U(x)=1$, $x \geq 0$ and $U(x)=0$, $x<0$. The initial condition can be prescribed as

$$\theta(x,y,0) = \theta_0 \tag{5}$$

III. Thermal Contact Resistance

The thermal contact resistance is defined as the ratio of the average temperature drop of the contact area to the total heat-transfer rate through the contact, i.e.,

$$R = \frac{1}{\Gamma}\left(\int_r [T(x,0,t) - T(x,c,t)]\mathrm{d}r \Big/ \int_r q\mathrm{d}r\right) \tag{6}$$

where Γ is the contact area. For the eccentric contact, $\Gamma = aL$ and the elemental area is $\mathrm{d}\Gamma = L\mathrm{d}x$. Equation (6) can be recast in dimensionless form as

$$R_c^* = \frac{1}{a(x)} \int_0^{a(x)} [\theta(x,0,Fo) - \theta(x,\alpha,Fo)]\mathrm{d}x \Big/ \int_0^{a(x)} q^* \mathrm{d}x \tag{7}$$

where $R_c^* = kLR$ is the dimensionless contact resistance.

IV. Analytical Solution

The expression for the thermal contact resistance R_c^* can be obtained by solving the boundary value problem for θ described in Sec. II and using Eq. (7). The solution for θ is obtained by the integral transform technique.[4] The Fourier cosine transforms with respect to the variables y and x are defined as

$$\overset{*}{\theta}(n,y,Fo) = \int_0^1 \theta(x,y,Fo)\cos n\pi x \mathrm{d}x \tag{8}$$

and

$$\overset{**}{\theta}(n,\mu,Fo) = \int_0^{\alpha} \overset{*}{\theta}(n,y,Fo)\cos(\mu y/\alpha)\mathrm{d}y \tag{9}$$

The inversion formulas can be written as

$$\theta(x,y,Fo) = \overset{*}{\theta}(0,y,Fo) + 2\sum_{n=1}^{\infty} \overset{*}{\theta}(n,y,Fo)\cos(n\pi x) \tag{10}$$

and

$$\overset{*}{\theta}(n,y,Fo) = \frac{2}{\alpha}\sum_{m=1}^{\infty} \frac{\mu_m}{\mu_m + \sin\mu_m \cos\mu_m} \cos\left(\frac{\mu_m y}{\alpha}\right) \overset{**}{\theta}(n,\mu_m,Fo) \tag{11}$$

Hence, μ_m, $(m=1,2,\ldots,\infty)$ are the roots of the characteristic equation

$$\mu \tan\mu = Bi\alpha \tag{12}$$

Simultaneous applications of the transforms [Eqs. (8) and (9)] over the system of equations (1-5) yield

$$\frac{\mathrm{d}\overset{**}{\theta}}{\mathrm{d}Fo} + \beta\overset{**}{\theta} = F(x) \tag{13}$$

where

$$F(x) = \int_{e-½a(x)}^{e+½a(x)} \frac{q^* \cos n\pi x}{b} \mathrm{d}x$$

and

$$\beta = n^2\pi^2 + \mu_m^2/\alpha^2 \tag{14}$$

The initial condition [Eq. (5)] becomes

$$\overset{**}{\theta}(n,\mu,0) = \theta_0(\alpha/\mu)\sin\mu, \qquad n=0$$
$$\overset{**}{\theta}(n,\mu,0) = 0, \qquad n \geq 1 \tag{15}$$

Equation (13) can be solved with the help of the initial condition [Eq. (15)]. The solution for $\overset{**}{\theta}(n,\mu,Fo)$ can be written as

$$\overset{**}{\theta}(n,\mu,Fo) = [\theta_0(\alpha/\mu)\sin\mu - F_1(x,0) + F_1(x,Fo)]e^{-\mu^2 Fo}, \qquad n=0$$
$$= [-F_1(x,0) + F_1(x,Fo)]e^{-\beta Fo}, \qquad n \geq 1 \tag{16}$$

where

$$F_1(x,Fo) = \int F(x,Fo)e^{\beta Fo}\mathrm{d}Fo$$

The expression for $\theta(x,y,Fo)$ can be written by applying the inversion equations (10) and (11). The final solution becomes

$$\theta(x,y,Fo) = \frac{2}{\alpha}\sum_{m=1}^{\infty} \frac{\mu_m \cos(\mu_m y/\alpha)}{\mu_m + \sin\mu_m\cos\mu_m}\left[\left(\frac{\alpha}{\mu_m}\right)\theta_0 \sin\mu_m - F_1(x,0) + F_1(x,Fo)\right]\exp\left(\frac{-\mu_m^2 Fo}{\alpha^2}\right)$$
$$+2\sum_{n=1}^{\infty}[F_1(x,Fo) - F_1(x,0)]\cos n\pi x \exp(-\beta_m Fo) \tag{17}$$

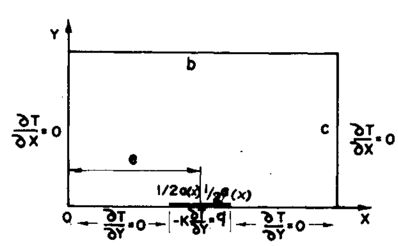

Fig. 1 Eccentric contact model.

V. Special Cases

Thermal constriction resistance as a function of Fourier number Fo can be expressed explicitly for different heat flux specifications, namely, $q^* = q_0 \exp(-PFo)$ and $q^* = q_0 \cos(\omega Fo + \epsilon)$. These cases are chosen because they are often encountered in a variety of design problems in space applications. Equation (17) is simplified for these cases and the corresponding thermal constriction resistances are calculated.

Case 1

$$q^* = q_0 \exp(-PFo) \tag{18}$$

In the case of an exponential heat flux prescribed on a contact length a, Eq. (17) reduces to

$$\theta(x,y,Fo) = \frac{2}{\alpha} \sum_{m=1}^{\infty} \frac{\mu_m \cos(\mu_m y/\alpha)}{\mu_m + \sin\mu_m \cos\mu_m} \times \left\{ \left(\frac{\alpha}{\mu_m}\right) \theta_0 \sin\mu_m \exp\left(\frac{-\mu_m^2 Fo}{\alpha^2}\right) - \frac{q_0 \cdot a}{\mu_m^2/\alpha^2 - P} \left[\exp\left(\frac{-\mu_m^2 Fo}{\alpha^2}\right) - \exp(-PFo) \right] - 4q_0 \sum_{n=1}^{\infty} \frac{1}{n\pi(\beta_{mn} - p)} \cos n\pi e \sin\frac{n\pi a}{2} \cos n\pi x \times [\exp(-\beta_{mn} Fo) - \exp(-PFo)] \right\} \tag{19}$$

For $q=0$, Eq. (19) reduces to a similar one obtained by Luikov (Ref. 4, p. 223, Eq. 6.3.29).

The thermal constriction resistance for $\theta_0 = 0$ (chosen arbitrarily) is expressed as

$$R_c^* = \frac{2}{\alpha} \sum_{m=1}^{\infty} \frac{\mu_m (1 - \cos\mu_m)}{\mu_m + \sin\mu_m \cos\mu_m} \left\{ \frac{1}{\mu_m^2/\alpha^2 - P} \times \left[\exp(-PFo) - \exp\left(\frac{-\mu_m^2 Fo}{\alpha^2}\right) \right] + 4 \sum_{n=1}^{\infty} \frac{1}{n^2 \pi^2 a (\beta_{mn} - P)} \sin n\pi a \times \sin\frac{n\pi a}{2} \cos n\pi e [\exp(-PFo) - \exp(-\beta_m Fo)] \right\} \tag{20}$$

Case 2

$$q^* = q_0 \cos(\omega Fo + \epsilon) \tag{21}$$

In the case of a periodic heat flux prescribed on a contact length a, the thermal constriction resistance for $\theta_0 = 0$ is expressed as

$$R_c^* = \frac{2}{\alpha} \sum_{m=1}^{\infty} \frac{\mu_m (1 - \cos\mu_m)}{\mu_m + \sin\mu_m \cos\mu_m} \left\{ \frac{1}{\mu_m^4/\alpha^4 + a\omega^2} \left[\left(\frac{\mu_m^2}{\alpha^2} \cos\epsilon + \omega \sin\epsilon \right) \exp\left(\frac{-\mu_m^2 Fo}{\alpha^2}\right) - \frac{\mu_m^2}{\alpha^2} \cos(\omega Fo + \epsilon) - \omega \sin(\omega Fo + \epsilon) \right] + 4 \sum_{n=1}^{\infty} \frac{1}{n^2 \pi^2 a (\beta_{mn}^2 + \omega^2)} \times \{ (\beta_{mn} \cos\epsilon + \omega \sin\epsilon) \exp(-\beta_{mn} Fo) - [\beta_{mn} \cos(\omega Fo + \epsilon) + \omega \sin(\omega Fo + \epsilon)] \} \sin n\pi a \cos n\pi e \sin\frac{n\pi a}{2} \right\} \tag{22}$$

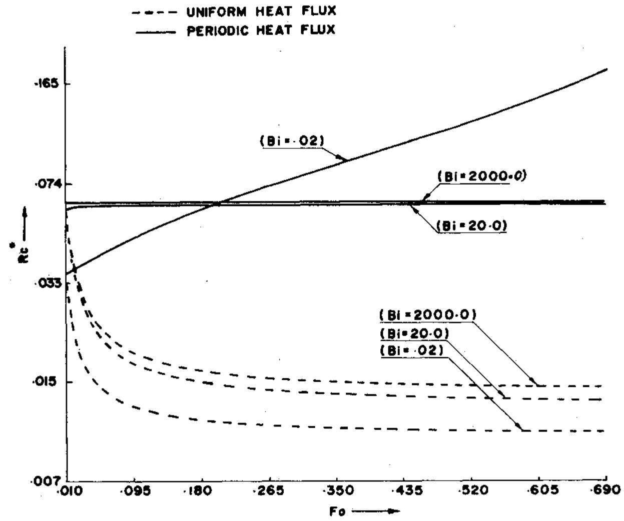

Fig. 2 Thermal constriction resistance as a function of Fo ($a = 0.1$, $e = 0.005$, $\alpha = 0.05$).

VI. Results

The thermal constriction resistance has been obtained for a range of Biot moduli $Bi = 0.01$-100 for a constant heat flux ($p = 0$) and for a periodic heat flux. The series evaluation was carried out for Eqs. (20) and (22). It is worth noting that the convergence of the series to the order of 10^{-5} is achieved by considering the first 20 terms of the series. The variation in the thermal constriction resistance R_c^* with the generalized time variable Fo for different Biot modules is shown in Fig. 2. Uniform variations in the thermal constriction resistance over generalized time are observed. The constriction resistance increases with the increase in the Biot modulus for a constant heat flux. For small values of Biot moduli, the heat flow is spread more uniformly over the upper convective surface as compared to the larger values of the moduli. Thus, for a given heat flux the temperature gradient from lower to the upper convective surface increases with the increase in the Biot modulus. As a result, the thermal constriction resistance increases as the Biot modulus increases. During the decreasing phase of a periodic heat flux, the constriction resistance increases with time for low Biot moduli, while it remains uniform for larger values. This reflects that for a poorly conducting solid/fluid interface, the temperature gradient between the two surfaces remains uniform although the heat flux decreases.

VII. Conclusion

The thermal constriction resistance in a convectively cooled plate is analyzed for an exponentially decreasing heat flux and a periodic heat flux. Unlike a periodic heat flux, an uniform variation in thermal constriction resistance due to variations in the Biot modulus is observed for a constant heat flux.

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